

Graphical representation of marginal and underlying probabilities in quantum mechanics

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Abstract

Wigner's marginal probability theory is revisited, and systematically applied to n -particle correlation measurements. A set of Bell inequalities whose corollaries are Hardy contradiction and its generalisation are derived with intuitive graphical analysis.

Keywords: Bell inequality, local reality, marginal probability

PACS: 03.65.Ud, 03.67.-a, 42.50.Ar

1. Introduction

An unambiguous signal of the departure from local causality in quantum mechanics is expressed as violation of Bell inequality [1] and its variants, the Bell theorems, such as CHSH inequalities [2], GHZ theorem [3] and Hardy's arguments [4, 5], which has been verified by the experiments by use of various physical systems [6, 7, 8, 9, 10]. The relations among these Bell theorems have been uncovered gradually [11, 12]. The violation of Bell inequality can be viewed also as a sign of the breakdown of the classical concepts of joint probability. As independently pointed out by Wigner and Fine [13, 14, 15], the Bell inequality, in its original form devised by Bell, presupposes underlying joint probability distribution for all possible measurement setups and outcomes in the experiment, whose marginal probabilities yield the probability distributions of the actual measurement setup of each run.

In this article, we revisit the argument by Wigner and Fine [13, 14, 15], and try to make clear the relation between their arguments and the concept of local reality. In the process, we uncover the graphic structure of the underlying joint probability, which we utilise to systematically to search general form of Hardy's equalities and associated inequalities.

This article is organized as follows. In Sec. 2, we introduce the concepts of the underlying probabilities which respects statistical independence and yield the measurable probability distributions as their marginal probabilities. In Sec. 3, we check consistency between the assumptions of statistical independence and the local reality. In Sec. 4, we turn to the graphical representation of the marginal probability for two-by-two experiment. Using this representation, we clarify that Hardy's equalities can be thought of as an extremal case of CHSH inequality. In Sec. 5, we generalise the above observation for multi-setting case and multi-partite case. Sec. 5 has the discussion and our conclusion.

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2. Marginal and Underlying Probabilities, Statistical Independence of Measurements

Consider a system of two $1/2$ -spins, each of which are measured separately by two observers Alice and Bob, both of whom can set up the measurement devices along two different axes, $a = 0, 1$ for Alice, and $b = 0, 1$ for Bob. We assume that the result of the dichotomic measurements is specified by $A = 0, 1$ for Alice, and $B = 0, 1$ for Bob. The usual sign notations of projection are recovered by identifications $s_A = (-)^A$ and $s_B = (-)^B$.

Let us denote the probability of Alice finding the system in the state A along the a -axis and Bob in B along b -axis by $P_{ab}(A, B)$. The assumption of statistical independence is expressed as

$$P_{ab}(A, B) = P_a(A)P_b(B). \quad (1)$$

Let us now consider parallel measurements on an ensemble of the system in two combinations of axes pair (a, b) and (a', b') . We write the joint probability of finding the results (A, B) and (A', B') as $W([A, B]_{ab}; [A', B']_{a'b'})$. If the parallel measurements are statistically independent, we should have

$$W([A, B]_{ab}; [A', B']_{a'b'}) = P_{ab}(A, B)P_{a'b'}(A', B'), \quad (2)$$

From the assumption (1), we have

$$P_{00}(A_0, B_0)P_{11}(A_1, B_1) = P_{01}(A_0, B_1)P_{10}(A_1, B_0) \quad (3)$$

since both are given by the product $P_0(A_0)P_1(A_1)P_0(B_0)P_1(B_1)$. This immediately leads to the equivalence between $W([A_0, B_0]_{00}; [A_1, B_1]_{11})$ and $W([A_0, B_1]_{01}; [A_1, B_0]_{10})$. This means that we can define unconditional underlying probability $\rho(A_0, B_0; A_1, B_1)$ which is the joint probability of Alice finding her system in state A_0 along 0-axis, A_1 along 1-axis and Bob finding his particle in state B_0 along 0-axis, B_1 along 1-axis;

$$\rho(A_0, A_1; B_0, B_1) = W([A_0, B_0]_{00}; [A_1, B_1]_{11}) = W([A_0, B_1]_{01}; [A_1, B_0]_{10}), \quad (4)$$

which signify the “reality” of the set of physical observables $(A_0, B_0; A_1, B_1)$ irrespective to the measurement. There are $2^4 = 16$ of U s. As probabilities, they are all non-negative real numbers, which add up to unity. The observable probabilities P_{ij} are obtained from ρ by partial sum as

$$\begin{aligned} P_{00}(A_0, B_0) &= \sum_{A_1, B_1} \rho(A_0, B_0; A_1, B_1), & P_{10}(A_1, B_0) &= \sum_{A_0, B_1} \rho(A_0, B_0; A_1, B_1), \\ P_{01}(A_0, B_1) &= \sum_{A_1, B_0} \rho(A_0, B_0; A_1, B_1), & P_{11}(A_1, B_1) &= \sum_{A_0, B_0} \rho(A_0, B_0; A_1, B_1). \end{aligned} \quad (5)$$

Following Wigner, we call the probability P_{abs} that are obtainable from direct measurements as *marginal probabilities*. Although the existence of unconditional probability ρ which is guaranteed by the relations (4), or equivalently (3), are derivable from the statistical independence between Alice’s and Bob’s observables, (1), the former is a looser assumption, from which the latter may not necessarily follow. So irrespective to the logic we have followed until now, we shall use the existence of unconditional probability U expressed in relations (4) and (5) as the basic assumption.

3. Derivation of Joint Probability Based on Locally Realistic Theory with Hidden Variable

Although in the derivation of $\rho(A_0, B_0; A_1, B_1)$, the cornerstone of our approach, we have pretended that it is derived from the statistical independence assumption of marginal probabilities, it can be derived by quite different set of assumption based on the concept of *hidden variable theory with local realism*. Of course, this is how the Bell inequality has been historically derived, and this is exactly the reason that Bell theorem is termed one of the most profound theorem in physics. We need, however to put it in perspective in view of the fact that there are vocal dissenting view [17] on the significance of Bell experiment that what is proven is just the statistical separability of probabilities, not the negation of local realism.

Here, we detail how Wigner has arrived at conditional and unconditional underlying probabilities W and ρ , stating from deterministic theory with hidden variables. Although we base our argument on the specific case of two-by-two Bell experiment, readers will see its generality and applicability to broader situations. We first want to construct a framework of deterministic theory with hidden variables, that are, unknown variables which are not to be observed directly, but whose ensemble average generates all $2^4 = 16$ marginal probabilities $P_{ab}(A, B)$, that completely specifies the outcomes of two-by-two Bell experiment.

It is, in fact, rather easy to have such a theory, given sufficient number of variables are brought in. Consider four variables q_{ab} with $a = 0, 1$ and $b = 0, 1$. we assume q_{ab} takes four values 0, 1, 2 and 3, which we express as

$$q_{ab} = A_{ab} + 2B_{ab} \quad (6)$$

with binary variables $A_{ab} = 0, 1$ and $B_{ab} = 0, 1$. We can alternatively think a *byte* variable λ made up of eight bits $\{A_{ab}, B_{ab}\}$, namely

$$\Lambda = \{A_{00}, B_{00}, A_{10}, B_{10}, A_{01}, B_{01}, A_{11}, B_{11}\} \quad (7)$$

as our hidden variables. These variables are governed by some unspecified dynamics. It is just sufficient if we accept that the result of the projection measurement with axes a for Alice and b for Bob is determined by the value of q_{ab} in such way that the projections s_A and s_B for Alice and Bob respectively are given by $s_A = (-)^{A_{ab}}$ and $s_B = (-)^{B_{ab}}$.

We now consider an ensemble of this deterministic system whose density of distribution on the space of Λ is given by

$$W(\Lambda) = W(A_{00}, B_{00}; A_{10}, B_{10}; A_{01}, B_{01}; A_{11}, B_{11}) \quad (8)$$

The observable probability P_{ab} are given by

$$\begin{aligned} P_{00}(A, B) &= \sum_{A_{10}, B_{10}} \sum_{A_{01}, B_{01}} \sum_{A_{11}, B_{11}} W(A, B; A_{10}, B_{10}; A_{01}, B_{01}; A_{11}, B_{11}), \\ P_{10}(A, B) &= \sum_{A_{00}, B_{00}} \sum_{A_{01}, B_{01}} \sum_{A_{11}, B_{11}} W(A_{00}, B_{00}; A, B; A_{01}, B_{01}; A_{11}, B_{11}), \quad \text{etc.}, \end{aligned} \quad (9)$$

and we should always be able to construct a deterministic theory whose initial ensemble of variable $\Lambda(t = 0)$, that corresponds to the initial setup of the experiment, evolve into the distribution $W(\Lambda)$ with $\Lambda = \Lambda(t)$ at the time of the measurement, that reproduces the observed $P_{ab}(A, B)$. It has to be remarked that the existence of this supposed underlying theory itself is rather unremarkable. The fact, that it requires an eight-bits variables capable of $2^8 = 256$ values to reproduce 16 experimental numbers, makes this theory an example of “cost-ineffective” generalization.

Now, we consider only those deterministic theories that respect local realism *à la* J.S.Bell. Then, the result of the Alice's measurement should not depend on *how Bob's measurement device is placed*, and Bob's result should not depend on *how Alice's measurement device is placed*. The values of the variables, in this type of theory, have to be limited in such way to reflect this fact, namely

$$A_{00} = A_{01} \equiv A_0, \quad A_{10} = A_{11} \equiv A_1, \quad B_{00} = B_{10} \equiv B_0, \quad B_{01} = B_{11} \equiv B_1. \quad (10)$$

So half of the bits in the variable Λ are redundant. In other word, local realistic theories underlying the observable $P_{ab}(A, B)$ is to be described by a *half byte* variable λ , which we define in the form

$$\lambda = \{A_0, B_0, A_1, B_1\}. \quad (11)$$

The ensemble-averaged system we observe in experiments is to be specified by the density of distribution

$$\rho(\lambda) = \rho(A_0, B_0; A_1, B_1), \quad (12)$$

which gives the observable probability P_{ab} in the form

$$P_{00}(A, B) = \sum_{A_1, B_1} \rho(A, B; A_1, B_1), \quad P_{10}(A, B) = \sum_{A_0, B_1} \rho(A_0, B; A, B_1), \quad \text{etc..} \quad (13)$$

Note that the variable λ can take 16 discrete values, as opposed to 256 for Λ . Consequently, there are 16 of $\rho(\lambda)$ functions as in contrast to 256 of $W(\Lambda)$ functions.

In formal term, the local realistic subclass of hidden variable theory is obtained from all possible theory by the reduction of density distribution

$$W(A_{00}, B_{00}; A_{10}, B_{10}; A_{01}, B_{01}; A_{11}, B_{11}) = \rho(A_{00}, B_{10}; A_{11}, B_{01}) \delta_{A_{00}, A_{01}} \delta_{A_{01}, A_{11}} \delta_{B_{00}, B_{10}} \delta_{B_{01}, B_{11}}, \quad (14)$$

or conversely,

$$\rho(A_0, B_0; A_1, B_1) = W(A_0, B_0; A_1, B_0; A_0, B_1; A_1, B_1). \quad (15)$$

We see that the “hidden variables” A_{ab} , B_{ab} and their reductions A_a , B_b are indeed hidden behind their guise as projection indices in W and U .

4. Diagrammatical Proof of Bell inequalities and Hardy Contradiction

The assumption of the existence of underlying unconditional probability $\rho(A_0, B_0; A_1, B_1)$ leads to several relations among marginal probabilities $P_{ab}(A, B)$.

Let us first reconfirm our notational conventions. The index a , that takes two values 0 and 1, stands for the axis of choice by Alice, along which the first particle is projectively measured to yield the value $s_A = +$ or $-$, which is alternatively expressed as $A = 0$ or 1 . Similarly, the index b , that can be 0 or 1, stands for the axis of Bob's measurement, which yields the value $s_B = +$ or $-$ that is alternatively expressed as $B = 0$ or 1 . We define inter-particle axis specific projections

$$i = A_0 + 2B_0, \quad j = A_1 + 2B_1. \quad (16)$$

The value of i represents the combined projection along the “0” axes of Alice and Bob, and j along their “1”- axes. We now reorder $\rho(A_0, A_1; B_0, B_1)$ by the indices i and j with the use of a matrix V defined by

$$V_{i,j} = \rho(A_0, B_0; A_1, B_1)|_{(i=A_0+2B_0, j=A_1+2B_1)}. \quad (17)$$

The partial sum (5) then becomes

$$\begin{aligned} P_{11}(A, B) &= \sum_{j=0}^3 V_{A+2B, j} & P_{10}(A, B) &= \sum_{i=0}^1 (V_{2B+i, A} + V_{2B+i, A+2}), \\ P_{01}(A, B) &= \sum_{j=0}^1 (V_{A, 2B+j} + V_{A+2, 2B+j}), & P_{11}(A, B) &= \sum_{i=0}^3 V_{i, A+2B} \end{aligned} \quad (18)$$

These expression has a very intuitive graphical representation Fig. 1. If we place ρ on a 4-by-4 grid according to the indices i and j , P_{00} are given by sums along horizontal lines, P_{11} by sums along vertical lines, while P_{10} and P_{01} are given by sums ribbon-shaped lines vertically and horizontally placed.

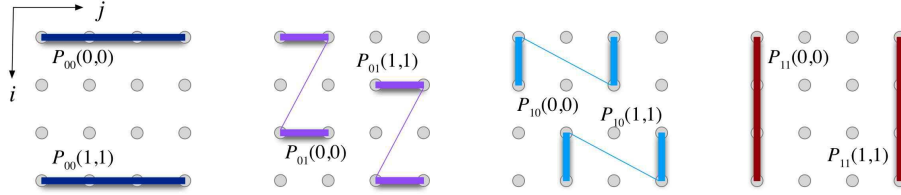


Figure 1: Marginal probabilities $P_{ab}(A, B)$ represented as a line or linked lines which cover the the grids that represent the underlying probabilities $V_{i,j}$ to be summed up.

From the compositions (18), it follows, for example, that

$$\begin{aligned} P_{10}(0, 0) + P_{01}(0, 0) + P_{11}(1, 1) &= 2V_{0,0} + V_{0,1} + V_{0,2} + V_{0,3} \\ &\quad + V_{1,0} + V_{1,2} + V_{1,3} \\ &\quad + V_{2,0} + V_{2,1} + V_{2,3} \\ &\quad + V_{3,3} \end{aligned} \quad (19)$$

which contains

$$P_{00}(0, 0) = V_{0,0} + V_{0,1} + V_{0,2} + V_{0,3} \quad (20)$$

and some more positive quantities. Thus we have a Bell inequality in the form

$$P_{10}(0, 0) + P_{01}(0, 0) + P_{11}(1, 1) - P_{00}(0, 0) \geq 0 \quad (21)$$

A graphical representation of this inequality is shown in Fig. 2.

A corollary immediately obtained is that

$$\text{if } P_{10}(0, 0) = P_{01}(0, 0) = P_{11}(1, 1) = 0 \text{ hold, then } P_{00}(0, 0) = 0 \text{ follows.} \quad (22)$$

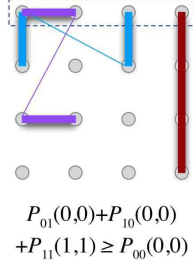


Figure 2: Graphical depiction of a Bell inequality among marginal probabilities $P_{ab}(A, B)$ represented as a line or linked lines for terms in LHS, a dashed box for the RHS, superimposed on the grid representing the underlying probabilities $V_{i,j}$ to be summed up.

This is nothing but the equality obtained by Lucien Hardy. A similar construction to (19), (20) leads to

$$\begin{aligned} P_{10}(1, 1) + P_{01}(1, 1) + P_{11}(0, 0) - P_{00}(1, 1) &\geq 0, \\ P_{10}(1, 0) + P_{01}(1, 0) + P_{11}(0, 1) - P_{00}(1, 0) &\geq 0, \\ P_{10}(0, 1) + P_{01}(0, 1) + P_{11}(1, 0) - P_{00}(0, 1) &\geq 0. \end{aligned} \quad (23)$$

With the definition of corelation C_{ab} ,

$$C_{ab} = P_{ab}(0, 0) - P_{ab}(1, 0) - P_{ab}(0, 1) + P_{ab}(1, 1) \quad (24)$$

inequalities (21) and (23) lead to

$$|C_{10} + C_{01} + C_{11} - C_{00}| \leq 2, \quad (25)$$

which is a celebrated CHSH inequality.

With the interchanges of axis indices a and b , and with interchanges of projections 0 and 1 for both A and B in $P_{ab}(A, B)$, there are 64 inequalities of the type (21). Each four of them sharing the same leg ab form a single CHSH inequality. As an obvious corollary, there are 64 variants of Hardy equality.

5. Three Particle Generalization and More

The extension to n -particle case is rather straightforward. We illustrate it with three particle case. Let us now consider a system consisting of three spin 1/2 particles that are respectively measured by Alice, Bob and Chris, all of whom have two choices each for the orientation of measurement represented by $a = 0$ or 1, $b = 0$ or 1 and $c = 0$ or 1, respectively. Alice's measured projection is represented by $A_a = 0$ or 1, Bob's by $B_b = 0$ or 1 and Chris' by $C_c = 0$ or 1. If there is an underlying probability $\rho(A_0, B_0, C_0; A_1, B_1, C_1)$, that presuppose the simultaneous existence of all projections for all three observers, the marginal probabilities P_{abc} which are the

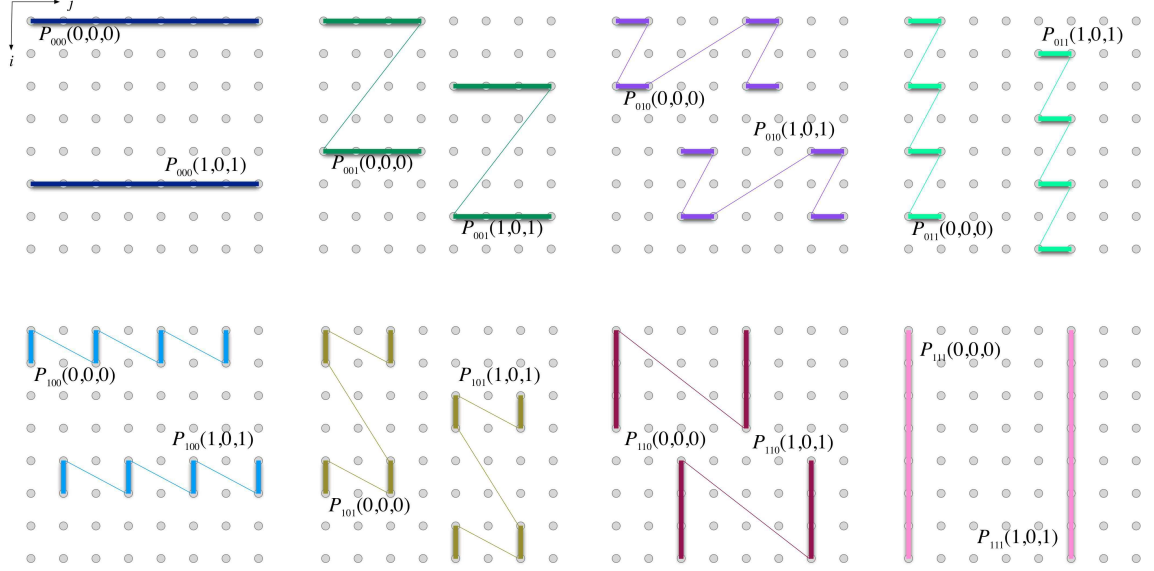


Figure 3: Graphical depiction of three-body marginal probabilities $P_{ab}(A, B, C)$ represented as a line or linked lines superimposed on the grids representing underlying probability $V_{i,j}$ to be summed up.

joint probabilities of direct observables are given by the partial sums

$$\begin{aligned}
 P_{000}(A, B, C) &= \sum_{A', B', C'} \rho(A, B, C; A', B', C'), & P_{100}(A, B, C) &= \sum_{A', B', C'} \rho(A, B, C; A', B', C'), \\
 P_{010}(A, B, C) &= \sum_{A', B', C'} \rho(A, B, C; A', B', C'), & P_{110}(A, B, C) &= \sum_{A', B', C'} \rho(A, B, C; A', B', C'), \\
 P_{001}(A, B, C) &= \sum_{A', B', C'} \rho(A, B, C; A', B', C'), & P_{101}(A, B, C) &= \sum_{A', B', C'} \rho(A, B, C; A', B', C'), \\
 P_{011}(A, B, C) &= \sum_{A', B', C'} \rho(A, B, C; A', B', C'), & P_{111}(A, B, C) &= \sum_{A', B', C'} \rho(A, B, C; A', B', C'). \quad (26)
 \end{aligned}$$

This set of construction may or may not result in the assumption of statistical independence

$$P_{abc}(A, B, C) = P_a(A)P_b(B)P_c(C), \quad (27)$$

although the latter necessarily results in the former. In case (27) holds, we have

$$\begin{aligned}
 P_{000}(A, B, C)P_{111}(A', B', C') &= P_{001}(A, B, C')P_{110}(A', B', C), \\
 P_{000}(A, B, C)P_{110}(A', B', C') &= P_{100}(A', B, C)P_{010}(A, B', C'), \\
 P_{111}(A, B, C)P_{001}(A', B', C') &= P_{011}(A', B, C)P_{101}(A, B', C'), \quad (28)
 \end{aligned}$$

etc.. With the three-digits grouping of indices

$$V_{i,j} = \rho(A_0, B_0, C_0; A_1, B_1, C_1) |_{(i=A_0+2B_0+4C_0, j=A_1+2B_1+4C_1)}, \quad (29)$$

the partial sums (26) become

$$\begin{aligned}
P_{000}(A, B, C) &= \sum_{j=0}^7 V_{A+2B+4C, j}, & P_{111}(A, B, C) &= \sum_{i=0}^7 V_{i, A+2B+4C}, \\
P_{110}(A, B, C) &= \sum_{i=0}^3 (V_{4C+i, A+2B} + V_{4C+i, A+2B+4}), & P_{001}(A, B, C) &= \sum_{j=0}^3 (V_{A+2B, 4C+j} + V_{A+2B+4, 4C+j}), \\
P_{100}(A, B, C) &= \sum_{i=0}^1 (V_{2B+4C+i, A} + V_{2B+4C+i, A+2} + V_{2B+4C+i, A+4} + V_{2B+4C+i, A+6}), \\
P_{011}(A, B, C) &= \sum_{j=0}^1 (V_{A, 2B+4C+j} + V_{A+2, 2B+4C+j} + V_{A+4, 2B+4C+j} + V_{A+6, 2B+4C+j}), \\
P_{010}(A, B, C) &= \sum_{j=0}^1 (V_{A+4C, 2B+j} + V_{A+4C+2, 2B+j} + V_{A+4C, 2B+4+j} + V_{A+4C+2, 2B+4+j}), \\
P_{101}(A, B, C) &= \sum_{i=0}^1 (V_{2B+i, A+4C} + V_{2B+i, A+4C+2} + V_{2B+4+i, A+4C} + V_{2B+4+i, A+4C+2}). \tag{30}
\end{aligned}$$

This extends the previous graphical expression of 4-by-4 matrix grid for two-particle case with 8-by-8 matrix grid on which ρ s are placed with indices $i = A_0 + 2B_0 + 4C_0$ and $j = A_1 + 2B_1 + 4C_1$. P_{000} and P_{111} are sums over horizontal and vertical lines, while P_{100} , P_{010} and P_{001} are sums over variously shaped ribbons, and P_{011} , P_{101} and P_{110} their respective mirror images with respect to the diagonal lines, all indicated in Fig.3. Comparison between Fig. 1 and Fig. 3 reveals how the $n = 3$ graphs are made out of $n = 2$ graphs; P_{000} , P_{100} and P_{010} are just the "sideway doubling" of P_{00} , P_{10} and P_{01} , and the remaining P_{111} , P_{011} and P_{101} are obtained by mirroring with respect to diagonal line that corresponds to the reversing axis indices ($a, b, c: 0 \leftrightarrow 1$). This construction carries over to any $n \rightarrow n + 1$ extension. Drawing various ribbons on the matrix grid as before, the following inequality is shown to hold

$$P_{100}(0, 0, 0) + P_{010}(0, 0, 0) + P_{001}(0, 0, 0) + P_{111}(1, 1, 1) - P_{000}(0, 0, 0) \geq 0 \tag{31}$$

whose corollary is a three body extension of Hardy equality, which states

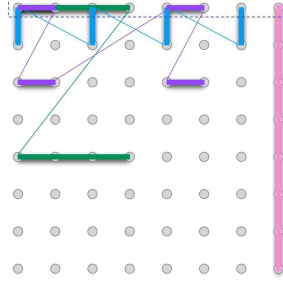
$$\begin{aligned}
&\text{if } P_{100}(0, 0, 0) = P_{010}(0, 0, 0) = P_{001}(0, 0, 0) = P_{111}(1, 1, 1) = 0 \text{ hold,} \\
&\text{then } P_{000}(0, 0, 0) = 0 \text{ follows.} \tag{32}
\end{aligned}$$

The geralization of this result

$$\begin{aligned}
&P_{100\dots 0}(0, 0, \dots, 0) + P_{010\dots 0}(0, 0, \dots, 0) + \dots + P_{00\dots 01}(0, 0, \dots, 0) \\
&\quad + P_{11\dots 1}(1, 1, \dots, 1) - P_{00\dots 0}(0, 0, \dots, 0) \geq 0, \tag{33}
\end{aligned}$$

and its associated Hardy equality is not hard to prove. Quite separately from above extension, we can prove a Zukowski inequality

$$C_{111} - C_{001} - C_{010} - C_{100} \geq -2 \tag{34}$$



$$P_{001}(0,0,0) + P_{010}(0,0,0) + P_{001}(0,0,0) + P_{111}(1,1,1) \geq P_{111}(0,0,0)$$

Figure 4: Graphical depiction of three-body marginal probabilities for the extended Hardy contradiction, represented as a line or linked lines for terms in LHS, a dashed box for the RHS, superimposed on the grid representing the underlying probabilities $V_{i,j}$ to be summed up.

in a similar fashion with the ribbons on 8-by-8 grid, whose corollary

$$\text{if } C_{001} = C_{010} = C_{100} = 1 \text{ then } C_{111} = 1 \quad (35)$$

is, of course, the negation of GHZ contradiction.

Another set of extension exists in the form of two party m -axes Bell experiments. Take for example, a system of two spin 1/2 particle each measured by Alice and Bob both of whom now has a choice of three projection axes $a = 0, 1, 2$ and $b = 0, 1, 2$. Experimental results are specified by 64 marginal probabilities $P_{ab}(A, B)$. The assumption of underlying unconditional probability distribution now involves 64 quantities $\rho(A_0, B_0; A_1, B_1; A_2, B_2)$. The proper tabulation of ρ now requires the definition of three two-digits indices

$$i_k = A_k + 2B_k, \quad k = 0, 2, 3, \quad (36)$$

thus is a cube grid of size 4-by-4-by-4. As depicted in Fig. 5, P_{kk} is the sum over a slice parallel to the cube surfaces, and P_{kl} with $k \neq l$ half sums of two non adjacent surfaces.

In a similar fashion to the previous cases, although now requiring some 3-dimensional recognition of patterns, we can prove a set of three-axes nonlocality inequalities. One such example is depicted in Fig. 6, showing an inequality

$$P_{00}(1, 1) + P_{10}(0, 0) + P_{02}(0, 0) - P_{12}(0, 0) \geq 0. \quad (37)$$

This is a three-axes version of inequalities of the type (21). In a sense, it can be regarded as physically identical to them, since just by renaming the axis "2" of Bob as "1", this simply reduces back to one of the two-axis type inequality. But this inequality does involve genuinely different three axes. We may obtain a three-axes version of Hardy type equality again by setting the first three term of (37) to be zero. Instead, we show another interesting face of this inequality by it requiring only the first term to be zero. We then have

$$\text{if } P_{00}(1, 1) = 0, \text{ then } P_{10}(0, 0) + P_{02}(0, 0) - P_{12}(0, 0) \geq 0. \quad (38)$$

This is nothing but the original Bell inequality in the form devised by J. S. Bell expressed in marginal probabilities instead of correlation functions, and the proof shown here is just the graphical dressing of Wigner's proof. Note that the original requirement of "being in singlet state" is

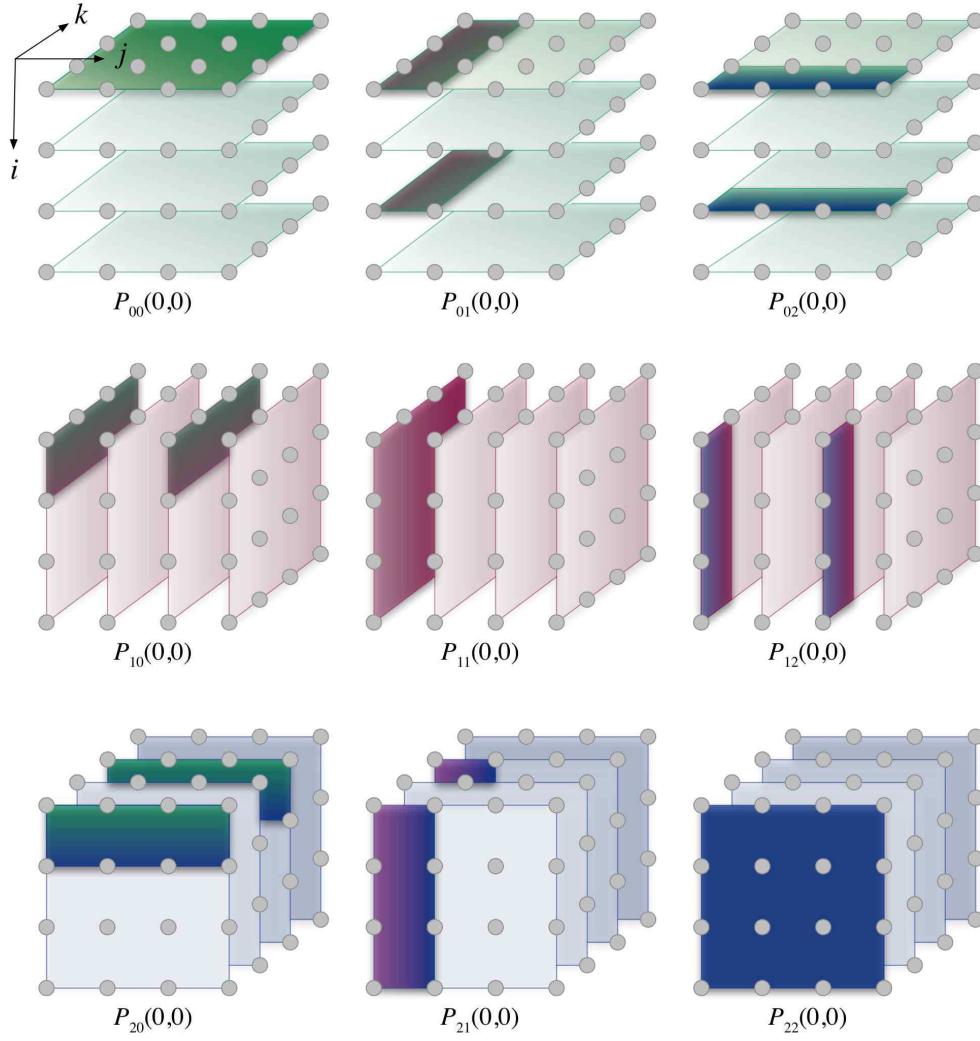
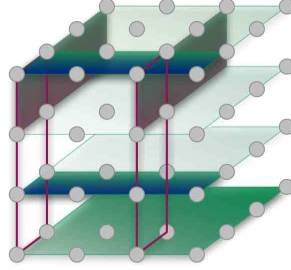


Figure 5: Graphical depiction of two-body three-axes marginal probabilities $P_{ab}(A, B)$ represented as planes painted in identical colors, which cover the three-dimensional grids that represent the underlying probabilities $V_{i,j,k}$ to be summed up.



$$P_{10}(0,0) + P_{02}(0,0) + P_{00}(1,1) \geq P_{12}(0,0)$$

Figure 6: Graphical depiction of two-body three-axes Bell inequality which leads to “Bell’s” Bell inequality. Marginal probabilities $P_{ab}(A, B)$ are represented as identically colored planes for terms in LHS, a transparent square for the RHS, which are superimposed on the grid representing the underlying probabilities $V_{i,j,k}$ to be summed up.

loosened to $P_{00}(1, 1) = 0$. We can now see that “Bell’s” Bell inequality occupies a midpoint between CHSH type inequality and Hardy type equality.

6. Discussion

The derivation of Bell inequalities with the use of underlying probabilities examined here is quite general. The argument is very straightforward, and this approach, in principle, is extendable to Bell inequalities with four or more choices and also to four or more players. But actual graphical representation quickly becomes messy and intractable for higher number of choices and players, and therefore, is not expected to be competitive against traditional systematic approaches [20]. There is, however, an advantage in our approach of having intuitive graphical representation, which should not be missed in pedagogical settings.

We thank Prof. I. Tsutsui and Dr. T. Ichikawa for stimulating discussions. This research was supported by the Japan Ministry of Education, Culture, Sports, Science and Technology under the Grant number 15K05216.

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